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**LETTER TO THE EDITOR**

# A note concerning quantum integrability

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**Abstract.** Heuristic arguments are presented supporting the conjecture that *almost* all quantum Hamiltonians are integrable in the sense that there exist $N$ ($N$ = number of freedoms) mutually commuting observables (which, in analogy with the classical action variables, can be chosen to be the number operators). This follows from perturbational considerations: the series may converge for almost all perturbations that preserve the discreteness of the spectra, because a 'quantum small denominator' almost always uniformly satisfies the condition of sufficient irrationality. The radius of convergence vanishes if $\hbar = 0$. The classical limit (as $\hbar \to 0$) of the quantum integrals of motion generically does not exist.

This is a speculative letter in the sense that I do not offer rigorous proofs, but I think that the result is correct. It concerns the important question of integrability in quantum mechanics and its preservation under small perturbations. As such, it is directed towards the search for the quantum analogy of the Kolmogorov–Arnold–Moser (KAM) theorem, but the expected result according to the announced conjecture is of course stronger: quantum Hamiltonians are almost always integrable, but the integrals of motion (the operators representing the observables) generically do *not* have the classical limit as $\hbar \to 0$. Our analysis is confined strictly to Hamiltonians with purely discrete spectra, and this imposes certain conditions upon the admissible perturbations: they must be bounded (see, e.g., Reed and Simon 1978). The perturbation series may converge for almost every admissible perturbation, as we will demonstrate, which is the basis for the conjecture that almost every quantum Hamiltonian is integrable. As we will see, the reason for convergence is that a quantum small denominator is a non-linear function of the integer indices (unlike the linear classical counterpart). The leading non-linear term depends on $\hbar$ and vanishes if $\hbar = 0$.

A quantum Hamiltonian $H_0$ with $N$ freedoms is defined *integrable* if there exist $N$ operators $A_n$, $1 \leq n \leq N$, $A_1 = H_0$, all of them functions of the coordinates $q_l$ and momenta $p_k$, i.e.

$$A_n = A_n(q_l, p_k) \qquad 1 \leq n \leq N \tag{1}$$

and such that all commutators vanish pairwise, i.e.

$$[A_l, A_k] = A_l A_k - A_k A_l = 0 \qquad 1 \leq l, k \leq N. \tag{2}$$

Of course, the coordinates and the momenta satisfy the canonical commutation relations

$$[q_l, p_k] = \mathrm{i}\hbar\delta_{lk}.$$

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We assume that if $H_0$ is a quantum integrable Hamiltonian, then there exists a unitary transformation which brings $H_0$ to the *normal form*

$$H_0 = H_0(N_1, \ldots, N_N) \tag{3}$$

where

$$N_n = z_n^+ z_n \qquad 1 \leqslant n \leqslant N \tag{4}$$

are the number operators, and

$$z_n^+ = (1/\sqrt{2})(q_n + \mathrm{i}p_n) \tag{5}$$

$$z_n = (1/\sqrt{2})(q_n - \mathrm{i}p_n) \qquad 1 \leqslant n \leqslant N \tag{6}$$

are the ladder operators. The number operators (4) are the quantum analogues of the classical action variables.

Now we consider the admissible perturbations of an integrable two-freedom ($N = 2$) Hamiltonian $H_0$. Let $H_0$ be in the normal form (3) and let the perturbed Hamiltonian $H$ be written as

$$H = H_0 + \varepsilon H_1 \tag{7}$$

where $\varepsilon$ is the perturbation parameter and $H_1$ is an analytic function of the ladder operators (5) and (6). It is an admissible perturbation represented generally by the power series

$$H_1 = \sum_{n,m=0}^{\infty} (z_1^{+n} z_2^{+m} f_{nm} + g_{nm} z_1^n z_2^m + z_1^{+n} h_{nm} z_2^m + z_2^{+n} l_{nm} z_1^m) \tag{8}$$

where the expansion 'coefficients' $f_{nm}$, $g_{nm}$, $h_{nm}$, $l_{nm}$ are functions of $N_1$ and $N_2$ only. In (8) we keep only the non-normal terms, the normal part of the perturbation being absorbed in $H_0$. The coefficients must satisfy the symmetry relations

$$f_{nm}^* = g_{nm} \qquad h_{nm}^* = l_{mn} \tag{9}$$

to ensure that $H_1$ is Hermitian, i.e. $H_1^+ = H_1$. The expansion (8) is precisely the analogue of the classical case, where the perturbation is expressed as a Fourier series in classical action-angle variables of $H_0$.

The central question is whether a unitary transformation $U$ exists, which restores the normal form for $H$. $U$ can be represented as

$$U = \exp(\mathrm{i}S) \tag{10}$$

$S$ being a Hermitian operator (a function of the ladder operators) written as a series

$$S = \sum_{n=1}^{\infty} \varepsilon^n S_n. \tag{11}$$

For $\varepsilon = 0$ $U$ is identity. Each $S_n$ is expanded in a similar manner to (8); for example

$$S_1 = \sum_{n,m=0}^{\infty} (z_1^{+n} z_2^{+m} a_{nm} + b_{nm} z_1^n z_2^m + z_1^{+n} c_{nm} z_2^m + z_2^{+n} d_{nm} z_1^m). \tag{12}$$

Since $S_1$ is Hermitian we again have

$$a_{nm}^* = b_{nm} \qquad c_{nm}^* = d_{mn}. \tag{13}$$

The question is thus whether there is an $S$ such that

$$H' = \exp(\mathrm{i}S) H \exp(-\mathrm{i}S) \tag{14}$$

is in normal form. Considering only the lowest (linear) term we find

$$H' = H_0 + \varepsilon H_1 - \mathrm{i}\varepsilon[H_0, S_1] + \mathrm{O}(\varepsilon^2) \tag{15}$$

so that $S_1$ must solve the equation

$$[H_0, S_1] = -\mathrm{i}H_1. \tag{16}$$

(As already mentioned, we assume $f_{00} = g_{00} = h_{00} = l_{00} = 0$.)

Before discussing a more general case we carefully calculate the solution of (16) for the special case of the quadratic $H_0$:

$$H_0 = \omega_1 N_1 + \omega_2 N_2 + \lambda N_1^2 + \mu N_1 N_2 + \nu N_2^2 \tag{17}$$

where $\omega_1$, $\omega_2$, $\lambda$, $\mu$, $\nu$ are real positive constants. We insert (12) into (16) and calculate all commutators. In doing so the following formulae (18)-(34) are used. They are derived in a straightforward manner:

$$[z_j, z_j^+] = \hbar \tag{18}$$

$$[z_j, z_j^{+m}] = m\hbar z_j^{+(m-1)} \tag{19}$$

$$[z_j^m, z_j^+] = m\hbar z_j^{(m-1)} \tag{20}$$

where $j = 1, 2$. Further,

$$[N_1, z_1^{+n}] = n\hbar z_1^{+n} \tag{21}$$

$$[N_1, z_1^m] = -m\hbar z_1^m \tag{22}$$

where $m$ and $n$ are arbitrary non-negative integers:

$$[N_1 N_2, z_1^{+n} z_2^{+m} a_{nm}] = z_1^{+n} z_2^{+m} a_{nm}(n\hbar N_2 + m\hbar N_1 + nm\hbar^2) \tag{23}$$

$$[N_1 N_2, b_{nm} z_1^n z_2^m] = -b_{nm}(n\hbar N_2 + m\hbar N_1 + nm\hbar^2) z_1^n z_2^m \tag{24}$$

$$[N_1 N_2, z_1^{+n} c_{nm} z_2^m] = z_1^{+n} c_{nm}(n\hbar N_2 - m\hbar N_1) z_2^m \tag{25}$$

$$[N_1 N_2, z_2^{+n} d_{nm} z_1^m] = z_2^{+n} d_{nm}(n\hbar N_1 - m\hbar N_2) z_1^n \tag{26}$$

$$[N_1^2, z_1^{+n} z_2^{+m} a_{nm}] = z_1^{+n} z_2^{+m} a_{nm}(2n\hbar N_1 + n^2\hbar^2) \tag{27}$$

$$[N_2^2, z_1^{+n} z_2^{+m} a_{nm}] = z_1^{+n} z_2^{+m} a_{nm}(2m\hbar N_2 + m^2\hbar^2) \tag{28}$$

$$[N_1^2, b_{nm} z_1^n z_2^m] = -b_{nm}(2n\hbar N_1 + n^2\hbar^2) z_1^n z_2^m \tag{29}$$

$$[N_2^2, b_{nm} z_1^n z_2^m] = -b_{nm}(2m\hbar N_2 + m^2\hbar^2) z_1^n z_2^m \tag{30}$$

$$[N_1^2, z_1^{+n} c_{nm} z_2^m] = z_1^{+n}(2n\hbar N_1 + n^2\hbar^2) c_{nm} z_2^m \tag{31}$$

$$[N_2^2, z_1^{+n} c_{nm} z_2^m] = -z_1^{+n}(2m\hbar N_2 + m^2\hbar^2) c_{nm} z_2^m \tag{32}$$

$$[N_1^2, z_2^{+n} d_{nm} z_1^m] = -z_2^{+n}(2m\hbar N_1 + m^2\hbar^2) d_{nm} z_1^m \tag{33}$$

$$[N_2^2, z_2^{+n} d_{nm} z_1^m] = z_2^{+n}(2n\hbar N_2 + n^2\hbar^2) d_{nm} z_1^m. \tag{34}$$

It can readily be verified that the comparison of coefficients in (16) yields

$$a_{nm} = (-\mathrm{i} f_{nm}/\hbar)[n\omega_1 + m\omega_2 + \lambda(2nN_1 + n^2\hbar) + \mu(nN_2 + mN_1 + nm\hbar) + \nu(2mN_2 + m^2\hbar)]^{-1} \tag{35}$$

$$c_{nm} = (-\mathrm{i} h_{nm}/\hbar)[n\omega_1 - m\omega_2 + \lambda(2nN_1 + n^2\hbar) + \mu(nN_2 - mN_1) - \nu(2mN_2 + m^2\hbar)]^{-1} \tag{36}$$

and that the relations (13) are satisfied if (9) is satisfied. We can rewrite (35) and (36) as follows:

$$a_{nm} = (-\mathrm{i}f_{nm}/\hbar)[(n\partial H_0/\partial N_1 + m\partial H_0/\partial N_2) + \hbar(\lambda n^2 + \mu nm + \nu m^2)]^{-1} \tag{37}$$

$$c_{nm} = (-\mathrm{i}h_{nm}/\hbar)[n\partial H_0/\partial N_1 - m\partial H_0/\partial N_2) + \hbar(\lambda n^2 - \nu m^2)]^{-1} \tag{38}$$

where $n$, $m = 0, 1, 2, \ldots$. The expressions $n\partial H_0/\partial N_1 \pm m\partial H_0/\partial N_2$ are exactly the classical small denominators. They occur in solving the classical analogue of (16). Note that they are linear in $n$ and $m$. The remainder of the denominator in (37) or (38) represents a quantum correction, linear in $\hbar$, but non-linear in $n$ and $m$.

If $H_0$ is not quadratic but a general analytic function of $N_1$ and $N_2$, then the results (37) and (38) generalise as follows:

$$a_{nm} = (-\mathrm{i}f_{nm}/\hbar)(F(n, m; N_1, N_2, \hbar))^{-1} \tag{39}$$

$$c_{nm} = (-\mathrm{i}h_{nm}/\hbar)(G(n, m; N_1, N_2, \hbar))^{-1} \tag{40}$$

where $F$ and $G$ are some analytic functions of the two variables $n$ and $m$ and depend on the parameters $N_1$, $N_2$ and $\hbar$. Of course, if $\hbar \to 0$ one has

$$F = n\partial H_0/\partial N_1 + m\partial H_0/\partial N_2 + \mathrm{O}(\hbar) \tag{41}$$

$$G = n\partial H_0/\partial N_1 - m\partial H_0/\partial N_2 + \mathrm{O}(\hbar). \tag{42}$$

The crucial point concerns the properties of the zero sets of $F(n, m)$ and $G(n, m)$. The curves of zero level are defined by

$$F(n, m) = 0 \qquad G(n, m) = 0. \tag{43}$$

If (43) has integer solutions, $n$, $m \in \mathbb{N}$, then the series (12) diverges. It is clear that for general analytic functions the probability of having integer solutions of (43) is zero. If there are no integer solutions, we must investigate how close a zero level curve can approach the integer lattice points. So let us assume that there are no integer solutions.

One important observation is that the non-linearities of $F$ and $G$ can give rise to closed zero level curves or, more generally, the zero level curve can have finite length. (In the quadratic case (37) the zero level curve is an ellipse.) In such a case the absolute value of the small denominator $|F|$, say, has a definite lower bound $F_{\mathrm{lb}} = F_{\mathrm{lb}}(N_1, N_2, \hbar) > 0$, but such that $F_{\mathrm{lb}} \to 0$ as $\hbar \to 0$. For any finite $N_1$, $N_2$ and $\hbar$ we find a uniform lower bound to the denominator of (39) so that the corresponding subseries of (12) converge for all finite $N_1$, $N_2$. This is already one important consequence of non-linearities.

If the zero level curve defined by $G = 0$, say, is not closed and has infinite length, then the behaviour of the small denominator $G$ in (40) is more critical. (In the quadratic case (38) $G = 0$ defines two hyperbolae.) For a general function $G(n, m)$ we would like to know the distances of the curve $G = 0$ from the lattice points $(n, m)$ with integer $n$, $m$ as $n$, $m \to \infty$.

There is one important thing to notice. In the quadratic case (37) and (38), for example, the behaviour of $G$ and $F$ for large $n$ and $m$ becomes independent of $N_1$ and $N_2$, but depends only on the (non-linearity) parameters $\mu$, $\nu$, $\lambda$ of $H_0$, and on $\hbar$. This property carries over to the arbitrary polynomial $H_0$. The crucial point is that, since $F$ and $G$ become independent of $N_1$ and $N_2$ in the limit $n$, $m \to \infty$, it is possible to satisfy the condition of sufficient irrationality *uniformly* for all $N_1$ and $N_2$. In other words, for almost any polynomial $H_0$ there exists a positive constant $K(H_0, \sigma, \hbar)$ depending only on the parameters of $H_0$, on $\hbar$ and on a real constant $\sigma > 0$, such that

$$|G(n, m)| \geqslant \frac{K(H_0, \sigma, \hbar)}{(m+n)^{1+\sigma}} \tag{44}$$

is satisfied for all $m$, $n \in \mathbb{N}$, $m+n \neq 0$ (similarly for $F$, of course). If (44) is obeyed, one can expect a method of superconvergence to be applicable uniformly for all $N_1$ and $N_2$, in contrast to the classical case, where the condition of sufficient irrationality is a function of the actions in the sense that $K$ depends on the actions. (It is further expected that (44) applies to almost every $H_0$ which is an analytic function of $N_1$ and $N_2$.) The uniform behaviour of the quantum small denominators as $n$, $m \to \infty$ is a consequence of the non-linearities of $H_0$ and leads us to the following conjecture.

*Conjecture.* There exists for almost any analytic integrable Hamiltonian $H_0$ a convergent (or superconvergent) perturbation series for $S$, such that the unitary transformation $U = \exp(\mathrm{i}S)$ restores the normal form of the perturbed Hamiltonian $H = H_0 + \varepsilon H_1$, where $H_1$ is an admissible perturbation.

If $H_1$ were not an admissible perturbation, it would prevent the series converging irrespective of other conditions. If true, the conjecture implies that almost all quantum Hamiltonians with purely discrete spectra are integrable, for they can be brought to the normal form $H'$ and thus each number operator $N_j$ is a constant of motion, i.e. it commutes with $H'$. However, the limit of the normal form $H'$ as $\hbar \to 0$ almost never exists: the convergence radius of the series vanishes if $\hbar = 0$, because the constant $K$ of (44) goes to infinity as $\hbar \to 0$, and the condition similar to (44) can no longer be satisfied uniformly for all finite $N_1$ and $N_2$, but only for certain selected values known from the KAM theorem.

Notice that the non-linearity of $H_0$ is decisive for the non-linear (and $\hbar$ dependent) corrections to the classical small denominators (see (37) and (38)). If $H_0$ is linear, we have the case of the Birkhoff–Gustavson normal form and its quantum equivalents (Robnik 1984, Ali 1985, Eckhardt 1986) and we are not surprised to face the divergence of the perturbation series in that case.

What has been (or will be) achieved when the unitary transformation $U = \exp(\mathrm{i}S)$ has been constructed? The meaning of the formalism so far is that we have found another Hamiltonian, namely the normal form $H' = UHU^{-1}$, which has the same spectrum as $H$, but it is still a function of the original variables $z_j$, $z_j^+$. Thus, $H'$ is an integrable Hamiltonian that is spectrally equivalent to $H$. However, in order to obtain an integrable Hamiltonian that is unitarily equivalent to $H(q_j, p_k)$ and has the same dynamics, we now have to transform the coordinates and momenta $q$ and $p$, or equivalently,

$$Z_j = U z_j U^+ \qquad Z_j^+ = U z_j^+ U^+ \tag{45}$$

where $U$ is a function of $z_j$, $z_j^+$. After the substitution $z_j = z_j(Z_k, Z_l^+)$, $z_j^+ = z_j^+(Z_k, Z_l^+)$ we finally obtain the Hamiltonian

$$H' = H'(Z_k, Z_l^+) \tag{46}$$

which is integrable and has the same spectrum and dynamics as $H$. In this way we see that the existence of $U$ guarantees that $H = H_0 + \varepsilon H_1$ is integrable. (By the assumption referred to at the beginning, $H'$ can be put in the normal form by an appropriate unitary transformation, since it is integrable.)

What matters is the complete unitary transformation of the quantum system $H$, namely the transformation of the Hamilton operator $H$ *and* the coordinates $q$ and momenta $p$. Only then is the dynamics preserved. As to the spectrum of $H$, we mention that there always exists an integrable Hamiltonian $H_0$, such that $f(H_0)$ is another integrable system that has the same spectrum as $H$, where $f$ is an analytic function, which is uniquely determined by $f(E_{0j}) = E_j$ for all $j = 1, 2, \ldots, \infty$. However, the transformation $f$ is not unitary, in general, and it does not preserve the dynamics (see also Robnik and Berry 1986).

In the appendix we briefly discuss some questions related to the quantum integrability.

The conclusion of this letter is that almost all quantum Hamiltonians with purely discrete spectra are (quantum) integrable, but that the classical limits of the integrals of motion generically do not exist. Further work is in progress to put these ideas on rigorous grounds.

## Appendix

We consider a quantum integrable system with classically ergodic limit. Consider the Wigner function

$$W_j(\boldsymbol{q}, \boldsymbol{p}, \hbar) = (\pi\hbar)^{-N} \int \psi_j(\boldsymbol{q}+\boldsymbol{x})\psi_j^*(\boldsymbol{q}-\boldsymbol{x}) \exp[-(2\mathrm{i}/\hbar)\boldsymbol{x}\cdot\boldsymbol{p}]\, \mathrm{d}^N\boldsymbol{x}$$

corresponding to the $j$th eigenstate $\psi_j(\boldsymbol{q})$. We make the following observations.

(i) As a consequence of the quantum integrability $W_j$ is localised at any finite $\hbar$, i.e. it is *not* ergodic, but concentrated in a certain region in phase space that might be called a '*quantum torus*'. It is clear that the semiclassical approach to the classically ergodic limit as $\hbar \to 0$ is manifested in the fact that the quantum torus fills the energy shell more and more densely as $\hbar \to 0$ or as $E_j \to \infty$.

(ii) $\psi_j(\boldsymbol{q})$ is *not* a Gaussian random function at any finite $\hbar$, but becomes such as $\hbar \to 0$, or as $E_j \to \infty$. The deviations from the Gaussian randomness for low lying *and* for high lying states should be observable and they have indeed been observed and analysed by Heller (1984), and recently evidence for such deviations was presented also by Berry and Robnik (1986).

(iii) It seems obvious that, as a consequence of the quantum integrability in systems with an ergodic classical limit, the only possible place for quantum tori to exist is near classically periodic orbits (all of them being unstable, of course). The phenomenon is described by Heller's theory of scars (Heller 1984), but appears here to be a consequence of quantum integrability.

We make two further remarks.

(iv) If the Hamiltonian has the normal form $H(N_1, N_2, \ldots, N_N)$ then, since it is quantum integrable, one might be puzzled to observe GOE or GUE energy level statistics (depending on the existence or non-existence of antiunitary symmetries—see Robnik and Berry (1986) and Robnik (1986)) rather than Poisson statistics. However, there is no paradox, for the contours $H(N_1, \ldots, N_N) = \text{constant}$ can have a non-trivial, $\hbar$-dependent and multiply connected topology. In addition, by the remark at the end of the main text, it is always possible to have a non-generic integrable system $f(H_0)$ such that it has any prescribed spectrum. Therefore it is not surprising to have quantum integrable systems with classically ergodic dynamics which display GOE or GOF statistics rather than Poisson statistics of energy levels.

(v) For a wide class of classically integrable systems it has been possible to construct the quantum integrals of motion (Hietarinta 1984).

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